Technical Notes

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On the Issue of Resonance in an Unsteady Supersonic Cascade

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IN a recent paper, Verdon and McCune[1] presented a linearized analysis of an unsteady supersonic cascade with subsonic axial velocity. It is an interesting advancement on Verdon's previous paper.[2] To achieve its objective of computing the pressure distribution, the analysis of Ref. 1 starts out to add the contributions from unsteady disturbances generated at all the oscillating airfoils below the reference airfoil in the cascade. The partial sums of the series were found to oscillate about apparent limiting values in general but, unfortunately, convergence was not proved. Aside from the problem of convergence, Ref. 1 also reports that under certain circumstances the numerical scheme broke down rather inexplicably. Thus it states in the concluding remarks that further work is needed to resolve these questions.

Several years ago, the author of the present Note encountered essentially the same series and found that it diverges at a certain number of discrete points, although these results were never published. The appearance of Ref. 1, therefore, seems a fitting opportunity to point out the divergence of the series, to offer it as a possible explanation for the aforementioned breakdown of the numerical scheme and to discuss its physical implications in regard to resonance and other salient points.

Consider, for example, the following kernel function $K(x)$, which appears in the integral equation (23) of Ref. 1:

$$K(x) = -(1/\mu)S(x)$$

where $S(x)$ is given by

$$S(x) = -\sum_{n=-\infty}^{-1} k\mu^2 n y_A e^{in\Omega} \times \frac{J_1\{k[(x-nx_A)^2-(\mu n y_A)^2]^{1/2}\}}{[(x-nx_A)^2-(\mu n y_A)^2]^{1/2}} \tag{1}$$

where x_A and y_A are the spatial distances between the adjacent airfoils in the cascade defined in Ref. 1, $\mu=(M^2-1)^{1/2}$ where M is the Mach number, $x_A-\mu y_A \geq 0$ (subsonic axial velocity), k is a frequency parameter, and Ω is related to the interblade phase lag σ. The nth term in the sum represents the influence of disturbances generated at the nth ($n<0$) below the first airfoil. We replace n by $-n$, rewrite J_1 in terms of the derivative of J_0 and obtain

$$S(x) = \sum_{n=1}^{\infty} e^{-in\Omega}\left[\frac{\partial J_0(W)}{\partial y}\right]_{y=0} \tag{2}$$

where

$$W(x,y) = k[(x+nx_A)^2-\mu^2(y+ny_A)^2]^{1/2} \tag{3}$$

We will show that $S(x)$ is divergent for certain combinations of parameters. For this, rewrite $W(x,y)$ as

$$W(x,y) = [z^2(n)-2z(n)Z(x,y)\cos\phi(x,y)+Z^2(x,y)]^{1/2} \tag{4}$$

where

$$z(n) = nk(x_A^2-\mu^2 y_A^2)^{1/2}$$

$$Z(x,y) = k(x^2-\mu^2y^2)^{1/2} \tag{5a}$$

$$\phi(x,y) = -i[\tanh^{-1}(\mu y/x) - \tanh^{-1}(\mu y_A/x_A)] - \pi \tag{5b}$$

This transformation, Eq. (5) may be conveniently achieved by introducing a set of auxiliary variables defined by $x=\rho\cosh\nu$, $\mu y=\rho\sinh\nu$, $x_A=\rho_A\cosh\nu_A$, and $\mu y_A=\rho_A\sinh\nu_A$. Now from Neumann's addition formula,[3] J_0 can be expressed as

$$J_0(W) = \sum_{m=0}^{\infty}\epsilon_m J_m(Z)J_m(z)\cos m\phi$$

where

$$\epsilon_m = 1 \text{ for } m=0 \tag{6a}$$

$$\epsilon_m = 2 \text{ for } m=1,2,\ldots \tag{6b}$$

for any complex values of Z, z, and ϕ. Substituting Eq. (6) into Eq. (2) and assuming the validity of the interchange of the order of differentiation and summation, we obtain

$$S(x) = \sum_{m=0}^{\infty}\epsilon_m \times\left\{\frac{\partial[J_m[Z(x,y)]\cos m\phi(x,y)]}{\partial y}\right\}_{y=0}\cdot a_m \tag{7}$$

where a_m is given by

$$a_m = \sum_{n=1}^{\infty} e^{-in\Omega}J_m[nk(x_A^2-\mu^2y_A^2)^{1/2}] \tag{8}$$

In deriving Eq. (7), we have made explicit use of the fact that $z(n)=nk(x_A^2-\mu^2y_A^2)^{1/2}$ which appears in Eq. (8), is independent of x and y, while Z and ϕ are independent of n. The series of a_m is called a Schlomlich series. Here we want to single out a_0 and examine its real part, which is given by

$$\mathrm{Re}\{a_0\} = \sum_{n=1}^{\infty}\cos(n\Omega)J_0[nk(x_A^2-\mu^2y_A^2)^{1/2}] \tag{9}$$

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Now from Ref. 4, for any t and x between 0 and π, the series

$$\sum_{n=1}^{\infty} \cos\ (nt) J_0(nx)$$

is divergent at $t=x$ and therefore Eq. (9) diverges at

$$\Omega = k(x_A^2 - \mu^2 y_A^2)^{1/2}$$

It is easy to observe that Eq. (9) has another singularity at

$$\Omega = -k(x_A^2 - \mu^2 y_A^2)^{1/2}$$

or more in general has an infinite number of singularities at

$$\Omega = \pm k(x_A^2 - \mu^2 y_A^2)^{1/2} + 2\pi n \quad n=0,\ \pm 1,\ \pm 2, \ldots$$

If we express Ω by the interblade phase angle σ, this becomes

$$\sigma + kMx_A = \pm k(x_A^2 - \mu^2 y_A^2)^{1/2} + 2\pi n \quad n=0,\ \pm 1,\ \pm 2, \ldots \tag{10}$$

At these points the series $S(x)$ diverges. We take special note that the divergent condition for $S(x)$ is independent of the value of x.

One can check the divergent condition, Eq. (10), more directly by the numerical evaluation of the original series Eq. (1) in the vicinity of divergence. Let the departure from the divergence be Δ, i.e.,

$$\sigma + kMx_A \pm k(x_A^2 - \mu^2 y_A^2)^{1/2} - 2\pi n = \Delta$$

where $\Delta = 0$ corresponds to the divergent condition, Eq. (10). The partial sum of Eq. (1) can then be written, after replacing n by $-\tilde{n}$, as

$$S_N(x) = k\mu^2 \sum_{\tilde{n}=1}^{N} \exp\{-i\tilde{n}[\pm k(x_A^2 - \mu^2 y_A^2)^{1/2} + \Delta]\} \tilde{n} y_A \frac{J_1\{k[(x+\tilde{n}x_A)^2 - (\mu \tilde{n} y_A)^2]^{1/2}\}}{[(x+\tilde{n}x_A)^2 - (\mu\tilde{n}y_A)^2]^{1/2}} \tag{11}$$

A number of numerical checks have been performed and they in fact confirm the divergence of the series at $\Delta = 0$. For example, for the cascade A of Ref. 1, the real and imaginary part of $S_N(x)$ (divided by $k\mu^2$) at $x=0.5$ are shown in Fig. 1 as functions of Δ for the various values of N [where the positive sign in the argument of the exponential function of Eq. (11) is chosen]. As can be observed immediately, the series tends to diverge at $\Delta = 0$. In addition, we note the following important points;

a) The divergence does not appear when N is 5 or 20. It starts to emerge at $N=200$ and becomes prominent at $N=2000$. In other words, the cause of the divergence is not the effect of the nearby airfoils but the cumulative influence of those airfoils located far from the reference airfoil.

b) As $\tilde{n}$ increases, the effect of x on the $\tilde{n}$th term in the series, Eq. (11), becomes insignificant.† When we combine this with a), it becomes evident why the divergent condition of Eq. (10) for $S(x)$ is independent of x.

c) For a given value of Δ away from the divergence condition, the partial sum $S_N(x)$ appears to oscillate as N increases. This behavior is in agreement with what was reported in Ref. 1.

Other series in Ref. 1 can also be shown to diverge. For instance, Eq. (20) of Ref. 1 contains a series, which, in the region downstream of the Mach wave emanating from the leading edge of the first airfoil, becomes

$$T(x,\eta,y) = \sum_{n=1}^{\infty} e^{-in\Omega} J_0\{k[(x-\eta+nx_A)^2 - \mu^2(y+ny_A)^2]^{1/2}\} \tag{12}$$

By a method exactly identical to that used for $S(x)$, we can readily show that $T(x,\eta,y)$ again becomes divergent at the condition, Eq. (10).

For the cascade A of Ref. 1, the linear relationship between σ and k as indicated in the divergent condition, Eq. (10) is plotted in Fig. 2 for several values of n. As mentioned, breakdown of the numerical scheme was reported in Ref. 1 for certain combinations of parameters; in Fig. 7 of Ref. 1, for a given value of k, σ was continuously increased starting from 0 and the failure occurred at a certain value of σ. When we add such combinations of σ and k closest to the breakdown situation to the present Fig. 2, where they are designated by circled points, breakdown is observed to occur near the divergent condition.

The present divergent criteria, Eq. (10), are identical with those obtained by Samoilovich.[5] He obtained it under the restricted condition corresponding to $x=\eta=y=0$ in Eq. (12). In such a case, the argument of J_0 becomes proportional to n and the use of the Poisson summation formula readily enables the series to be transformed into another series representation, which possesses the singular points at Eq. (10). In the present derivation, the divergent condition has been obtained for the more general case of any finite values of x, η, and y.

It is a matter of considerable interest to observe that Eq. (10) is formally the same as the resonance condition in a subsonic cascade,[6] and therefore it may be given the physical meaning similar to that discussed in Ref. 7. Thus, for the present divergence of the series associated with the supersonic cascade, Samoilovich also gave the physical interpretation of "resonance." Contrary to the subsonic cascade, it is, however, highly unlikely that resonance at these conditions could indeed occur for a supersonic cascade. The reason is as follows. We have found that the divergence is the direct consequence of the cumulative contribution of those airfoils located far from the reference airfoil; in Fig. 1, the peak does not appear when the number of the preceding airfoils N is 5 or 20, and it starts to emerge only at $N=200$. Needless to say, such an effect is computed within the framework of linearized formulation. It is well known, however, that the linearized treatment of a supersonic flow breaks down in the field far from an airfoil. According to the linear theory, disturbances created by the airfoil would propagate unattenuated, even to infinity, for both the low- and high-frequency limits (for high-frequency behavior, see Ref. 8.) Although in the near field the linearized theory is a good approximation, the contribution of the nonlinear terms become no longer negligible in the far field and there it encroaches on and modifies the effect predicted by the linear theory. Physically, this follows from the fact that, by the time disturbance reaches the far field, two nonlinear effects ignored in the acoustic theory—the convection of disturbance by the *local* and *instantaneous* fluid velocity and its propagation at the nonuniform speed of sound—have cumulatively taken their toll and distorted the shape of the wavelet given by the linearized theory. Thus, as Lighthill[9] puts it: "... the failure of linearized theory ... is explained by the fact that ... while yielding adequate results in a limited region, may yield a worse and worse approximation to the solution farther and farther from where the boundary conditions determining the solution were applied." Consequently, only the influence of a limited numbers of airfoils ($N \simeq 5$, say, for a typical cascade) in the neighborhood of the reference airfoils can accurately be predicted by the linearized theory. As the distance from the reference airfoils increases, the nonlinear effect would rapidly alter the linearized con-

†This enables one to check the divergent condition directly by using the asymptotic formula of J_1 in Eq. (11).

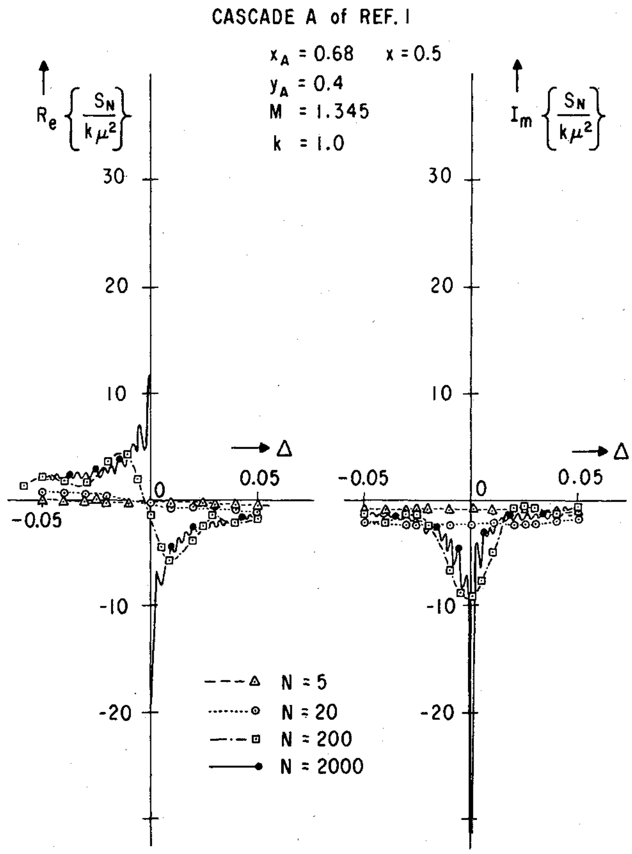

Fig. 1 Divergence phenomena.

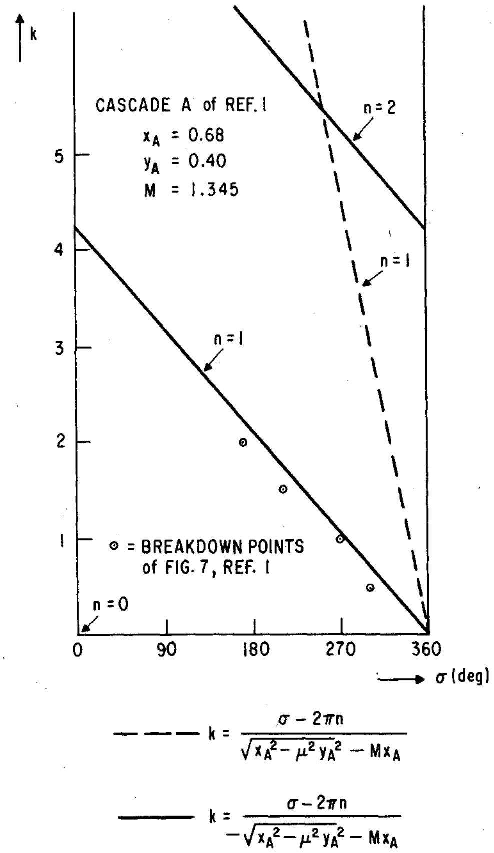

Fig. 2 $\sigma - k$ relationship.

tribution whose *pro forma* sum over a large number of airfoils is responsible for "resonance." These considerations therefore cast a serious doubt on the prospect of resonance at the present divergent condition in an actual supersonic cascade, unless evidence to the contrary is found. (When the Mach number is increased in the present subsonic leading edge problem, the solution, including the behavior at 'resonance,' should approach the lowest limit of the supersonic leading edge problem of Ref. 10. Since the latter does not possess any such 'resonance,' this appears to provide additional evidence to the contrary.)

Even when the sum is convergent, the aforementioned failure of the supersonic linearized theory in the far field raises the obvious question on the very scheme of the summation over an infinite array of cascaded airfoils within the framework of linearized analysis. We hasten to add, however, that this does *not* mean that one cannot solve the supersonic cascade problem by linearized analysis. It is indeed feasible to accomplish this by using a different formulation adopted in Ref. 11. The formulation may be called a "passage approach" and attention is focused on the reference passage between two blades. In place of the infinite cascade arrangement, we impost an equivalent periodicity requirement that the flow at any given point in the reference passage be the same as the flow at the corresponding point in the adjacent passage (with the exception of an interblade phase lag). Since the problem is set up within the blade passage and, furthermore, because of the limited domain of dependence of a point in a supersonic flow, it turns out that the formulation involves only the near field where the linearized theory is valid. This approach leads to a closed-form expression for the pressure distribution, correct to the linear order of the frequency. The analysis yields the so-called unique incidence effect as the steady limit, agrees with the sonic limit of the supersonic leading edge analysis, and correctly reflects the effect of the back pressure. The expression does not suffer breakdown at Eq. (10) and this is one more reason to preclude resonance under these conditions‡ in a supersonic cascade.

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‡The expressions of Ref. 11 possess a singularity at $\sigma = 0$, which is related to the unique incidence effect.